Partial Differential Equations I

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[1] A satisfactory treatment of the theory of partial differential equations requires, as bare minimum, an understanding of graduate level analysis. This is not to say that the undergraduate can't glean anything useful from a first look into the field, but we will not have the toolset or machinery to prove a handful of the theorems that follow. In any case, they shall be presented and deployed for our use in the qualitative and quantitative investigations of partial differential equations. Unlike with ordinary differential equations, finding an explicit solution to a PDE is the exception rather than the norm. Chapter 2 deals with first-order PDEs which can all be solved for closed-form solutions. After this point, however, entire chapters will be devoted to the solution and analysis of select second-order equations. In particular, we will analyze the big three second-order PDEs: Chapters 4 and 5 will cover the wave equation, Chapter 8 the heat equation, and Chapter 10 Laplace's equation. It may be shocking that there is so much to be said about individual linear second-order equations, but it is this difference in richness that separates the ordinary from the partial.

Contents

[1]The content of these notes is largely based on *Partial Differential Equations: A First Course* by Rustum Choksi.

1 Basic Definitions

A partial differential equation is an equation relating a function and any of its derivatives. These notes will cover

- the modeling and derivation of important PDEs,
- some methods of (explicit) solution to PDEs (Fourier series + transform),
- the uniqueness and stability of solutions to PDEs,
- the behavior of solutions to PDEs.

1.1 Notation

Partial derivatives of $u = u(x_1, x_2, ..., x_n)$ will be written $u_{x_i} = \frac{\partial u}{\partial x_i}$. Higher order derivatives will be written $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. The gradient of u is written $\nabla u = (u_{x_1}, u_{x_2}, ..., u_{x_n})$.

1.2 Some well-known/important PDEs

1. Let $u = u(x_1, x_2, x_3, t)$ be a function of three spatial variables and time. The PDE

$$u_t = \kappa(u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}) + Q(x_1, x_2, x_3, t) \tag{1}$$

   is known as the **diffusion (heat) equation**. This equation governs temperature and diffusion processes such as pollution in the air. The term

$$\Delta u = \nabla^2 u = u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}$$

   is called the Laplacian of u, and the function Q is referred to as the heat source.

2. The equilibrium solution to a diffusion process occurs when u no longer changes in time. Letting $u_t = 0$ in the diffusion equation above yields

$$\Delta u = Q \tag{2}$$

   which is known as **Poisson's equation**. When the heat source is identically zero, the resulting equation

$$\Delta u = 0 \tag{3}$$

   is known as **Laplace's equation**.

3. Let $u = u(x_1, x_2, x_3, t)$ be a function of three spatial variables and time. The PDE

$$u_{tt} = c^2 \Delta u \tag{4}$$

is known as the **wave equation**. In one spatial dimension, the function $u(x_1, t)$ may represent a vibrating string; in two dimensions, $u(x_1, x_2, t)$ may represent surface waves in a pond; and in three dimensions, $u(x_1, x_2, x_3, t)$ may represent acoustic waves.

Each of the above PDEs is second-order, that is they involve partial derivatives of maximum order two. The above equations are also all linear. Every PDE can be rewritten by separating terms that contain u and its derivatives from terms that contain exclusively the independent variables:

$$\mathcal{L}(u) = f(x_1, x_2, ...).$$

For instance, the diffusion equation can be rewritten as

$$u_t - \kappa \Delta u = Q(x_1, x_2, x_3, t)$$

where $\mathcal{L}(u) = u_t - \kappa \Delta u$ and $f(x_1, x_2, x_3, t) = Q(x_1, x_2, x_3, t)$. For a given PDE, $\mathcal{L}$ is called the differential operator and the PDE is said to be linear if $\mathcal{L}$ is linear in u. This amounts to $\mathcal{L}$ satisfying the following two properties for any two sufficiently smooth u_1 and u_2 and c real:

(a) $\mathcal{L}(u_1 + u_2) = \mathcal{L}(u_1) + \mathcal{L}(u_2)$,

(b) $\mathcal{L}(cu) = c\mathcal{L}(u)$.

By sufficiently smooth, we mean $u \in C^k$ where k is the order of the PDE.

4. Let $x \in \mathbb{R}$ and v be a given function of x. The **simple transport equation** is

$$u_t + v(x) u_x = 0. \tag{5}$$

This is a first-order linear PDE.

5. It will often be more efficient to write the independent spatial variables as a vector $\boldsymbol{x} = (x_1, x_2, x_3)$. For a complex-valued function $u = u(\boldsymbol{x}, t) = u(x_1, x_2, x_3, t)$, the **Schrodinger equation**

$$i u_t = -\Delta u + V(\boldsymbol{x}) u \tag{6}$$

is a second-order linear equation where $V(\boldsymbol{x})$ is a potential.

6. The first-order nonlinear PDE

$$|\nabla S(\boldsymbol{x})|^2 = \eta(\boldsymbol{x}) \tag{7}$$

is known as the **eikonal equation** where $\eta(\boldsymbol{x})$ is the index of refraction. The unknown function $S(\boldsymbol{x}) = S(x_1, x_2, x_3)$ is a scalar function of three variables, so its gradient is

$$\nabla S(\boldsymbol{x}) = (S_{x_1}, S_{x_2}, S_{x_3}).$$

Writing out the LHS reveals that the PDE is, in fact, nonlinear:

$$(S_{x_1})^2 + (S_{x_2})^2 + (S_{x_3})^2 = \eta(\boldsymbol{x}).$$

1.3 Review: Some Facts from Calculus

Recall that for a multivariate function u, the gradient of u is

$$\nabla u(\boldsymbol{x}) = \nabla u(x_1, ..., x_n) = (u_{x_1}, ..., u_{x_n}),$$

and is the vector that points in the direction of steepest ascent at $\boldsymbol{x}$. For any given unit vector $v \in \mathbb{R}^n$, the directional derivative of u in the direction v is

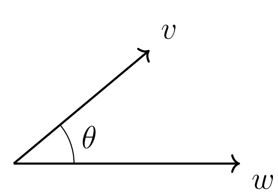
$$\nabla u(\boldsymbol{x}) \cdot v = u_{x_1} v^{(1)} + ... + u_{x_n} v^{(n)}.$$

Sometimes we write $\nabla u \cdot v = D_v u = \partial_v u$. This scalar quantity gives the rate of change of the function u in the direction of v. The dot product above is computed using component-wise multiplication, but a separate calculation will yield the same result: given two vectors v and w in $\mathbb{R}^n$, the dot product is also found by

$$v \cdot w = |v||w| \cos\theta$$

where θ is the angle between the two vectors in $\mathbb{R}^n$ and the magnitude of the vector v is

$$|v| = (v^{(1)2} + ... + v^{(n)2})^{1/2}.$$



Calc fact: Suppose $u(x,t)$ is continuously differentiable, with x and t being real variables. Then

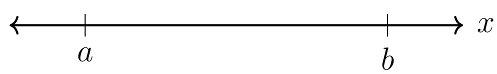
$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b u_t(x,t)\,dx.$$

This procedure is known as **differentiating under the integral** and is only valid if u_t exists and is continuous.

2 First-Order PDEs and the Method of Characteristics

2.1 The Simple Transport Equation

Think of the x-axis as a sort of river or channel, capable of transporting a material along it. Let $[a,b]$ denote any interval along this x-axis.



We will consider a density function $u(x,t)$ for the material in the interval. At any given time t, this function tells you how much of the material is packed at point x and is measured in units of “stuff”/length. By fixing time, we may find the total amount of material inside the interval by integrating the density with respect to x from a to b:

$$\text{amount of material in } [a,b] = \int_a^b u(x,t)\,dx \quad \text{at time } t.$$

By letting time vary, we can find the rate of change of the amount of material in the interval by differentiating this integral with respect to time:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx.$$

The amount of material in the interval changes only when stuff enters or leaves the interval. To measure the movement of material, let’s let $\phi(x,t)$ be the flux of material at location x and time t, measured in units of “stuff”/time. This measures how much of the material is traveling through a point at any given time. The flux at a certain point is positive if the material is moving from left to right, negative if the material is moving right to left, and zero if there is no movement. The only way for material to enter or leave the interval is through the endpoints, so that at any given time the rate of change of material in the interval is equal to the flux on the left face minus the flux on the right face. Mathematically, we have

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(a,t) - \phi(b,t).$$

By bringing the time derivative inside the integral (we'll assume a solution exists and is continuously differentiable) and rewriting the RHS as a second integral, we arrive at

$$\int_a^b u_t(x,t)\,dx = -\int_a^b \phi_x(x,t)\,dx$$

which is equivalent to

$$\int_a^b (u_t + \phi_x)\,dx = 0.$$

Since the interval $[a, b]$ is arbitrary, it must be the case that $u_t + \phi_x = 0$ for all x and all $t > 0$. This PDE is known as the **convection (or simple transport) equation**.

To attach a physical interpretation to this equation, let the x-axis denote a river flowing with a given velocity $v(x,t)$ measured in miles/hr. Suppose the material under consideration is logs. Then the density u is measured in logs/mile and gives the concentration of logs at a certain location and time. The flux ϕ, measured in logs/hr, is the amount of logs flowing through a point at a certain instant, so it is the product of the density and velocity:

$$\phi(x,t) = v(x,t)u(x,t).$$

Under this interpretation, we may substitute our ϕ into the simple transport equation to arrive at

$$u_t + (vu)_x = 0,$$

or

$$u_t + vu_x + v_x u = 0.$$

This is a first-order linear PDE. Let's solve it!

Example. Suppose $v = v_0 \in \mathbb{R}$ is constant and consider the convection equation

$$u_t + v_0 u_x = 0. \tag{8}$$

This can be rewritten using a dot product as

$$(u_x, u_t) \cdot (v_0, 1) = 0$$

which states that the directional derivative of u in the direction $(v_0, 1)$ is zero. Because this derivative is zero, it does not matter that $(v_0, 1)$ may not be a unit vector (as is usually important for directional derivatives). What does this dot product tell us? It states that in the $xt-$plane, the rate of change of u along lines parallel to the vector $(v_0, 1)$ is zero; that is to say u is constant along these lines.

At each point (x, t) in the upper half-plane (which is our domain of inquiry since time is non-negative), the solution u can be sampled at that point to get $u(x, t)$. Along any line parallel to the vector $(v_0, 1)$, that solution is constant. This constant can (and will) vary between lines, but if we find the value of the solution for just one point on every line, we will know the solution everywhere (since the lines fully fill the upper-half plane). These lines are called the **characteristic curves** of the PDE. Let's solve this PDE subject to the initial condition

$$u(x, 0) = g(x) \quad \text{for } -\infty < x < \infty.$$

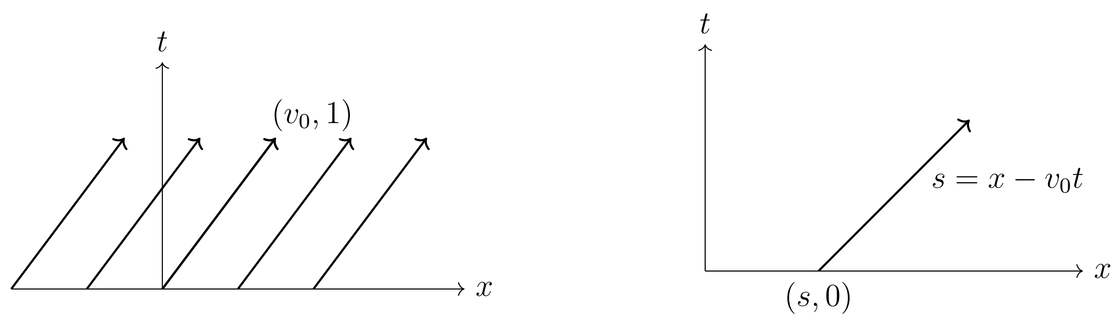


Figure 1: Characteristic lines in the $xt-$plane. Each line is characterized by its x-intercept. The solution u is constant along the line $s = x - v_0 t$, hence it is a function of the intercept alone: $u = g(s)$.

In the figure above, characteristics are plotted in the $xt-$plane. Notice that they are all parallel to the vector $(v_0, 1)$. In the right figure, a single characteristic is considered. Knowing the value of the solution at just one point on this line allows us to know it everywhere on the line. But we already know the value of the solution at the x-intercept of this line since we are given initial data! Indeed, if we denote by s the x-intercept, then at this point $u(s, 0) = g(s)$. To find the equation for the line, note that

$$\frac{dt}{dx} = \text{slope} = \frac{1}{v_0}$$

so that the line passing through the intercept $(s, 0)$ has equation

$$t = \frac{1}{v_0}(x - s).$$

Different choices of the x-intercept s fully characterize different lines (since they are all parallel and thus never intersect), meaning that s can be taken as a parameter. The solution depends solely on the value of s alone. Due to this, the solution to the PDE is

$$u(x,t) = g(s) = g(x - v_0 t).$$

Another way to think about it is this: we would like points that fall on the same characteristic line to be mapped by u to the same output. If two points (x_1, t_1) and (x_2, t_2) lie on the same characteristic, then $x_1 - v_0 t_1 = s = x_2 - v_0 t_2$ and our choice of u, caring about s alone, maps these points to the same output, namely $g(s)$.

It is always good practice to check that your function is indeed a solution to the PDE. The function clearly satisfies the initial data. To show it satisfies the PDE, we compute

$$\begin{aligned} u_x &= g'(x - v_0 t) \cdot 1 \\ u_t &= g'(x - v_0 t) \cdot (-v_0). \end{aligned}$$

From this, we have

$$u_t + v_0 u_x = -v_0 \cdot g'(x - v_0 t) + v_0 \cdot g'(x - v_0 t) = 0$$

and we are done.

There is a geometric reason as to why this PDE is known as the simple transport equation. At time $t = 0$, we have that the solution is $u_0 = g(x)$. We can think of this as an initial wave or signal. At time $t = 1$, the solution is $u_1 = g(x - v_0)$, which is the initial wave translated v_0 units to the right; at time $t = 2$, the solution is $u_2 = g(x - 2v_0)$, which is the initial wave translated $2 \cdot v_0$ units to the right; and so on. As suggested, we can think of this transport PDE as moving an initial wave or signal horizontally along the x-axis at a speed of v_0 units per unit of t.

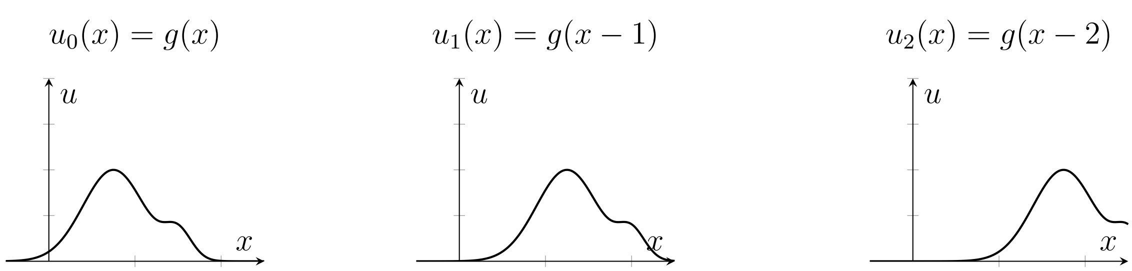


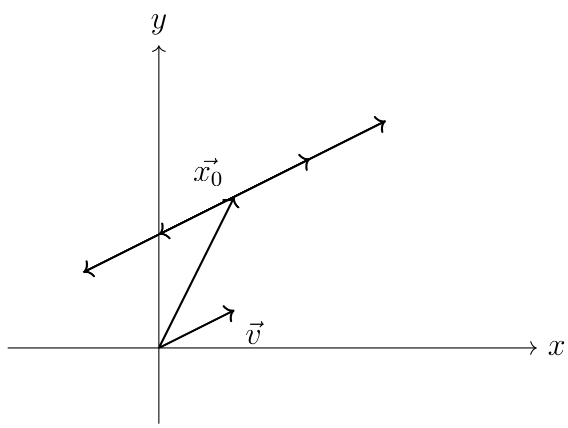
Figure 2: Solution of the transport equation $u_t + u_x = 0$ at various times. The initial wave $g(x)$ propagates to the right with speed $v_0 = 1$ without changing shape.

2.2 Review: Parametric Representation of Curves

The method of characteristics relies upon the computation of directional derivatives along curves. Going forward, these curves will be parametrically defined. This section provides a review of some of familiar parametric curves.

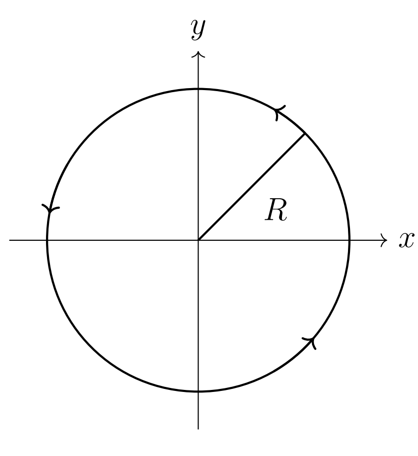
1. **Planar lines:** To parametrize a curve in the plane, we need two functions, each of a parameter s, for the two coordinates of the curve. As the parameter s takes on different values, the functions $(x(s), y(s))$ will trace out the curve. We require two pieces of information to parametrize a line: a point $\vec{x_0} = (x_0, y_0)$ on the line and a vector $\vec{v}$ parallel to the line. With these known, the parametric equations for the line are

$$(x(s), y(s)) = \vec{x_0} + s\vec{v} \qquad -\infty < s < \infty.$$



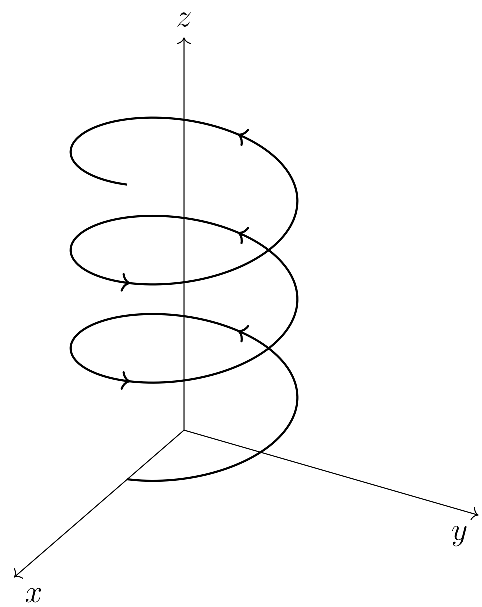
2. **Planar circles:** To parametrize a circle centered at the origin with radius R, we use

$$x(s) = R\cos(s)$$
$$y(s) = R\sin(s).$$



3. **3D helix:** For a helix in $\mathbb{R}^3$, the x and y coordinates are exactly as for a planar circle, but the z coordinate is simply identified with the parameter s:

$$\begin{aligned} x(s) &= R\cos(s) \\ y(s) &= R\sin(s) \\ z(s) &= s. \end{aligned}$$



2.3 The Method of Characteristics

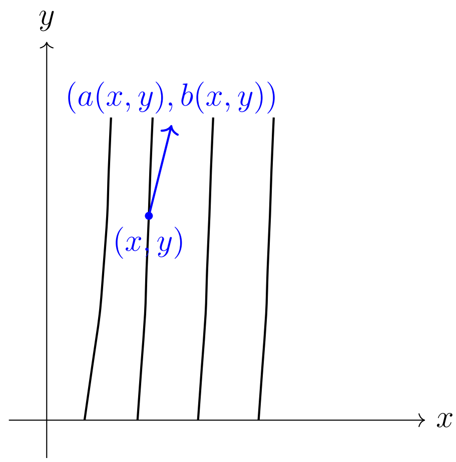
We are ready to formalize the method used to solve the simple transport equation earlier. The method involves degenerating a PDE into a series of ODEs along special curves called **characteristics**. These ODEs can be solved for each curve which, when "strung together," create a tapestry that fills the domain and provides us with a solution everywhere in the domain. To begin, consider a general first-order linear PDE for a function $u = u(x, y)$:

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y). \tag{9}$$

The idea is to try and convert this PDE into an ODE. We can rewrite this equation as

$$\nabla u(x, y) \cdot (a(x, y), b(x, y)) + c(x, y)u = d(x, y) \tag{10}$$

where the dot product is the directional derivative of u at (x, y) in the direction (a, b).



Example. We'll look at an easy example first. Let a and b be constant with $c = d = 0$. Then our PDE becomes the simpler

$$au_x + bu_y = 0. \tag{11}$$

Suppose we are given some auxiliary data along the x-axis

$$u(x, 0) = f(x).$$

Rewriting the PDE to be a directional derivative produces

$$\nabla u \cdot (a, b) = 0$$

which means, as always, that the solution u is constant along lines parallel to the vector (a, b) (equivalently, u is constant along lines with slope b/a). These lines are referred to as the characteristics of the PDE. Let's fix any one line pointing in the direction (a, b):

$$t \mapsto (x(t), y(t)).$$

We will forgo writing an explicit parametric equation for this line since it is to be taken general; the only condition, of course, is that its slope be b/a. To this end, we know the derivatives of both the x and y coordinates in terms of the parameter t:

$$\frac{dx}{dt} = a,$$
$$\frac{dy}{dt} = b.$$

In fact, we can compute the derivative of u with respect to the parameter t via the chain rule:

$$\frac{d}{dt}u(x(t), y(t)) = u_x\frac{dx}{dt} + u_y\frac{dy}{dt} = au_x + bu_y = 0.$$

This should make perfect sense: if $(x(t), y(t))$ is a line with slope b/a, then the solution u is constant everywhere along this curve. Consequently, the derivative of u with respect to the parameter t is null.

Now we choose to introduce a *second* parameter to the equations for x and y. While t serves to move us along a characteristic (where u is kept constant), the new parameter s will move us *between* different characteristics (where u will vary). The combining effects of having two parameters mean the entire domain will be visited for select choices of s and t. So now, write $x = x(s,t)$, $y = y(s,t)$, and use s to parametrize the initial curve, which in this case is the x-axis:

$$x(s,0) = s,$$
$$y(s,0) = 0,$$
$$u(s,0) = f(s).$$

These are the three initial conditions for the ODEs we found above. Written again here, these equations form what is called the **characteristic system**:

$$\frac{\partial x}{\partial t} = a,$$
$$\frac{\partial y}{\partial t} = b,$$
$$\frac{\partial u}{\partial t} = 0.$$

Let's solve this system of three ODEs!

$$\frac{\partial x}{\partial t} = a \implies x(s,t) = at + c_1(s)$$
$$x(s,0) = c_1(s) = s \implies \boxed{x = at + s}$$

$$\frac{\partial y}{\partial t} = b \implies y(s,t) = bt + c_2(s)$$
$$y(s,0) = c_2(s) = 0 \implies \boxed{y = bt}$$

$$\frac{\partial u}{\partial t} = 0 \implies u(s,t) = c_3(s)$$
$$u(s,0) = c_3(s) = f(s) \implies \boxed{u = f(s)}$$

Now return to x and y as independent variables

$$t = \frac{1}{b}y,$$
$$s = x - at = x - \frac{a}{b}y$$

to reveal a final solution of

$$u(x, y) = f(s) = f(x - \frac{a}{b}y).$$

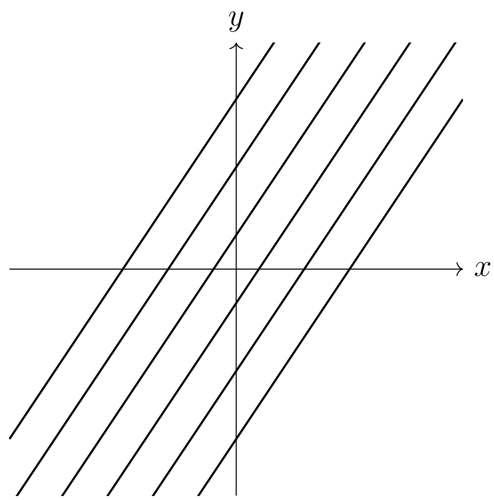


Figure 3: Characteristic lines with slope b/a

Example. Consider the following PDE and initial condition:

$$u_x + 2u_y + u = 0$$
$$u(x, 0) = \sin x.$$

Now that we know the general line of attack, we can work considerably faster. Begin by parameterizing the independent variables: $x = x(s, t)$ and $y = y(s, t)$. Rewriting the PDE as

$$u_x + 2u_y = -u$$

exposes the hidden chain rule

$$u_t = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t} = u_x + 2u_y = -u.$$

This proposes the following characteristic system:

$$\frac{\partial x}{\partial t} = 1 \qquad x(s, 0) = s,$$
$$\frac{\partial y}{\partial t} = 2 \qquad y(s, 0) = 0,$$
$$\frac{\partial u}{\partial t} = -u \qquad u(s, 0) = \sin s.$$

Solving the first two of these subject to their initial conditions gives us the parametric equations

$$y(s,t) = 2t \implies t = \frac{y}{2},$$
$$x(s,t) = t + s \implies s = x - \frac{y}{2}.$$

The third differential equation involves a routine separation of variables

$$\frac{du}{dt} = -u \implies \frac{1}{u}\,du = -\,dt \implies \int \frac{1}{u}\,du = -\int dt$$
$$\implies \ln u = -t + c \implies u = Ce^{-t}.$$

Now we can make use of the initial condition:

$$u(s,t) = C(s)e^{-t} \implies u(s,0) = C(s)e^{0} = C(s) = \sin s$$
$$\implies u(s,t) = (\sin s)e^{-t}.$$

Alas, with x and y in their rightful places as independent variables, we are left with

$$u(x,y) = (\sin s)e^{-t} = \sin(x - \frac{y}{2})e^{-y/2}.$$

2.4 Review: Solution Methods to Various ODEs

It has been made evident that the method of characteristics rests upon our ability to solve ordinary differential equations. This section will serve to review some of the more common solution methods to ODEs.

1. **Separable first-order:** We seek a solution to a differential equation of the form

   $$f(u)u'(x) = g(x)$$

   where $u = u(x)$ is the unknown function. Suppose we have anti-derivatives for f and g, that is, let F satisfy $\frac{dF}{du} = f(u)$ and G satisfy $\frac{dG}{dx} = g(x)$. Our ODE may be rewritten as

   $$\frac{dF}{du}\frac{du}{dx} = \frac{dG}{dx} \implies \frac{d}{dx}(F(u(x)) = \frac{dG}{dx}$$

   where we have made use of the chain rule to compress the LHS. The functions $F(u(x))$ and $G(x)$ have the same derivative with respect to x; what we can conclude is that these functions are the same up to an additive constant:

   $$F(u) = G(x) + C.$$

   This an implicit solution for the function $u(x)$.

2. **First-order linear:** Consider a general first-order linear ODE

$$u'(x) + p(x)u(x) = g(x).$$

The trick is to multiply through by an "integrating factor"

$$\mu(x) = e^{\int p(x)dx}.$$

Doing so produces

$$u'\mu + up\mu = \mu g.$$

Because $\mu' = p\mu$, a hidden product rule reveals our solution:

$$(u\mu)' = \mu g \implies u\mu = \int \mu g \, dx + C \implies u = \frac{1}{\mu} \int \mu g \, dx + \frac{C}{\mu}.$$

3. **Second-order, constant coefficient, linear, homogeneous:** The differential equation in question is

$$au'' + bu' + cu = 0$$

where a, b, and c are real constants. Like any good solution method, a subtle trick is involved. We make an a priori "guess" to the solution

$$u = e^{rx}$$

for some, as of now unknown, r. Plugging this into the ODE and dividing by $e^{rx} > 0$ yields

$$ar^2 + br + c = 0$$

whose roots we will call r_1 and r_2. If these two roots are real and not equal to one another, then the solution to the differential equation is

$$u = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

If, however, the roots are not real, then they are complex conjugates

$$r_1, r_2 = \alpha \pm i\beta$$

and the general solution is

$$u = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$

2.5 Some Harder Examples

Let's recall once more the method of characteristics, this time in a more general sense. A PDE is quasilinear if the coefficients on all terms of highest order depend only on the independent variables and u. Suppose we are given a first-order quasilinear PDE in x and y

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

subject to an auxiliary condition

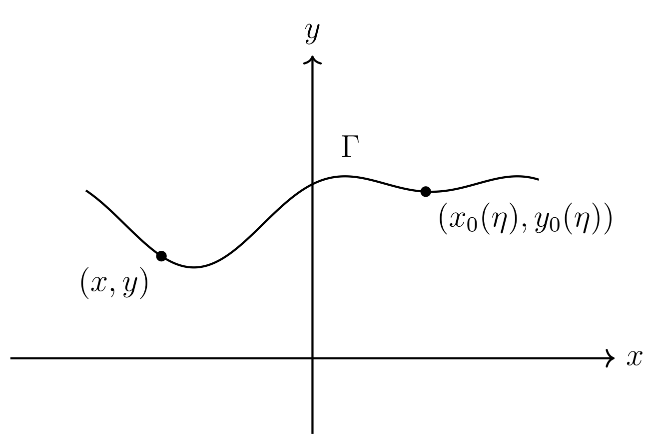
$$u = f(x, y) \quad \text{for } (x, y) \in \Gamma,$$

where Γ is a curve in our domain. A simple two-step procedure summarizes the method.

I. **Write down the characteristic system.** Recall that the independent variables x and y are doubly-parametrized so that the we can both pick out certain characteristics from their intersection with the initial curve Γ and follow along these characteristics. To prevent the bad practice of blindly following notation, we will change the names of the parameters from the last example. However, we will keep the convention that the first parameter (now η) chooses the characteristic, while the second (now s) traverses it. So we will write $x(\eta, s)$, $y(\eta, s)$, and $u(\eta, s)$ as the unknowns of the characteristic system

$$\begin{aligned} \frac{\partial x}{\partial s} &= a(x, y, u) \qquad & x(\eta, 0) &= x_0(\eta), \\ \frac{\partial y}{\partial s} &= b(x, y, u) \qquad & y(\eta, 0) &= y_0(\eta), \\ \frac{\partial u}{\partial t} &= c(x, y, u) \qquad & u(\eta, 0) &= f(x_0(\eta), y_0(\eta)). \end{aligned}$$

II. **Invert the variables to find $s(x, y)$ and $\eta(x, y)$.** With this accomplished, the solution u can be written as a function of x and y.



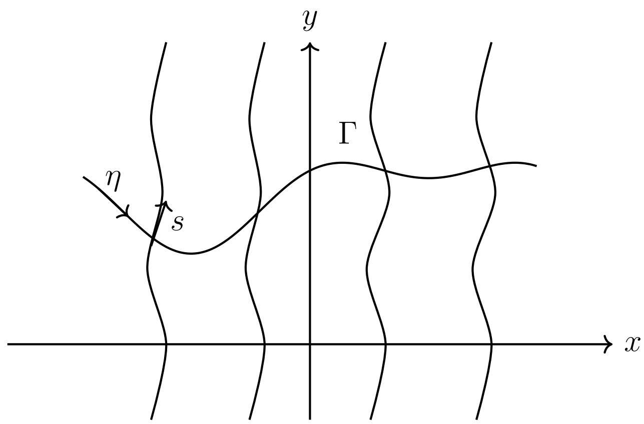


Figure 4: Top: Initial curve Γ parametrized by η. Bottom: Characteristics intersecting Γ parametrized by s.

Example. We'll solve the PDE and auxiliary condition

$$u_x - u_y + 2yu = y$$
$$u(x,0) = x.$$

As always, we'll set up the characteristic system:

$$\frac{\partial x}{\partial s} = 1 \qquad x(\eta,0) = \eta,$$
$$\frac{\partial y}{\partial s} = -1 \qquad y(\eta,0) = 0,$$
$$\frac{\partial u}{\partial s} = -2yu + y \qquad u(\eta,0) = \eta.$$

This is our first instance of coupling among the ODEs: notice the presence of y in the equation for u. Let's begin solving these

$$x(\eta,s) = s + c_1(\eta)$$
$$x(\eta,0) = \eta \implies \boxed{x = s + \eta}$$
$$y(\eta,s) = -s + c_2(\eta)$$
$$y(\eta,0) = 0 \implies \boxed{y = -s}.$$

With $y = -s$ now known, we can replace the instance of y in the third ODE to get

$$\frac{\partial u}{\partial s} = (-2)(-s)u - s \implies u_s - 2su = -s$$

which is a first-order linear differential equation. Since the coefficient on u is $p(s) = -2s$, our integrating factor is

$$\mu(s) = e^{\int p(s)ds} = e^{\int -2s\,ds} = e^{-s^2}.$$

Multiplying by the integrating factor and collapsing the product rule gives

$$\frac{\partial}{\partial s}(e^{-s^2}u) = -se^{-s^2} \implies e^{-s^2}u = \int -se^{-s^2}\,ds.$$

A simple change of variables $\tau = -s^2$, $d\tau = -2s\,ds$, produces

$$e^{-s^2}u = \frac{1}{2}\int e^{\tau}\,d\tau = \frac{1}{2}e^{-s^2} + c_3(\eta) \implies u = \frac{1}{2} + c_3(\eta)e^{s^2}.$$

Making use of our auxiliary condition allows us to find $c_3(\eta)$

$$u(\eta, 0) = \eta = \frac{1}{2} + c_3(\eta)e^0 \implies c_3(\eta) = \eta - \frac{1}{2}$$

which enables us to write the expression for u in terms of the parameters:

$$u(\eta, s) = \frac{1}{2} + (\eta - \frac{1}{2})e^{s^2}.$$

As our final step, we invert the parametric relations and solve for u in terms of x and y:

$$\begin{aligned} s &= -y, \\ \eta &= x - s = x + y, \\ u(x, y) &= \frac{1}{2} + (x + y - \frac{1}{2})e^{y^2}. \end{aligned}$$

As a final check of our understanding, we might ask ourselves what the characteristics are. Because the coefficients belonging to u_x and u_y are constant, we are led to believe they're lines. More specifically, they are lines of slope -1 as is evident by the equation relating η (the characteristic-picking parameter) and the variables x and y:

$$y = -x + \eta.$$

Example. An example of nonlinear characteristics:

$$\begin{aligned} &u_t + \frac{2}{1+t^2}u_x = x, \\ &u(x, 0) = \cos^2 x \quad \text{for } -\infty < x < \infty. \end{aligned}$$

The characteristic system is

$$\begin{aligned} \frac{\partial x}{\partial s} &= \frac{2}{1+t^2} & x(\eta, 0) &= \eta, \\ \frac{\partial t}{\partial s} &= 1 & y(\eta, 0) &= 0, \\ \frac{\partial u}{\partial s} &= x & u(\eta, 0) &= \cos^2 \eta. \end{aligned}$$

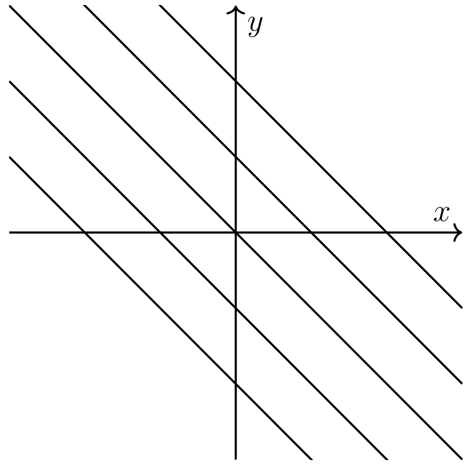


Figure 5: Characteristics with slope −1

All three of these are coupled, so let's start with the easiest, the equation for t:

$$t = s + c_1(\eta),$$
$$t(\eta, 0) = c_1(\eta) = 0 \implies \boxed{t = s}.$$

Now our parameter s is really just the variable t, so the ODE for x can be solved directly for x as a function of t:

$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial t} = \frac{2}{1+t^2} \implies x = \int \frac{2}{1+t^2}\,dt \implies x = 2\arctan t + c_2(\eta),$$
$$x(\eta, 0) = 2\arctan 0 + c_2(\eta) = c_2(\eta) = \eta \implies \boxed{x = 2\arctan t + \eta}.$$

The third ODE for u can be solved by identifying s with t and x with the expression in t found above:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} = 2\arctan t + \eta \implies u = 2\int \arctan t\,dt + \eta t + c_3(\eta).$$

The integral of arctan can be found using the integration by parts formula

$$\int vw' = vw - \int wv'$$

with $v = \arctan t$ and $w' = 1$:

$$\int \arctan t\,dt = t\arctan t - \int \frac{t}{1+t^2}\,dt = t\arctan t - \frac{1}{2}\ln(1+t^2).$$

Returning to our formula for u, we have

$$u = 2t\arctan t - \ln(1+t^2) + \eta t + c_3(\eta),$$
$$u(\eta, 0) = c_3(\eta) = \cos^2\eta \implies u(\eta, t) = 2t\arctan t - \ln(1+t^2) + \eta t + \cos^2\eta.$$

Finally, plug in $\eta = x - 2\arctan t$ to reveal

$$u(x,t) = 2t\arctan t - \ln(1+t^2) + (x - 2\arctan t)t + \cos^2(x - 2\arctan t).$$

The characteristics are found via the relationship between the variables x and t and the parameter η:

$$x = 2\arctan t + \eta.$$

Inverting this relation to have x on the horizontal axis gives us

$$t = \tan\left(\frac{x-\eta}{2}\right).$$

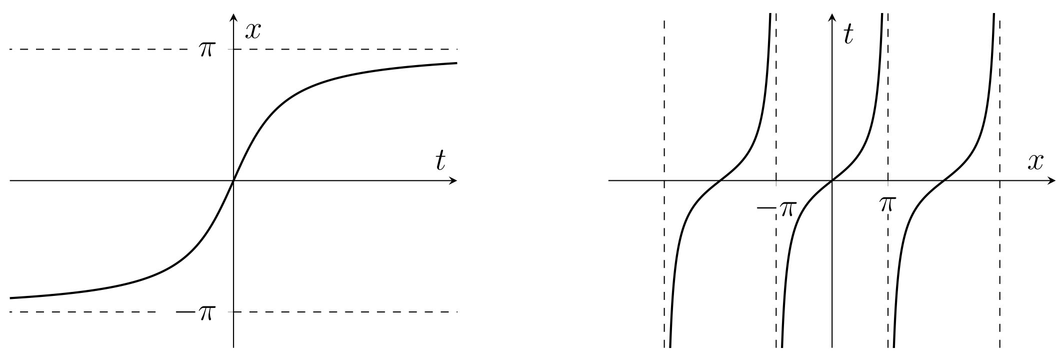


Figure 6: Left: $x = 2\arctan t$. Right: $t = \tan((x-\eta)/2)$ for $\eta = -2\pi, 0, 2\pi$.

Example. Here's another example of nonlinear characteristics. This will happen, as you can probably guess, when the coefficients attached to the gradient components are non-constant:

$$yu_x - xu_y = 1,$$
$$u = 2 \text{ along } y = x.$$

The corresponding characteristic system is

$$\frac{\partial x}{\partial s} = y \qquad x(\eta,0) = \eta,$$
$$\frac{\partial y}{\partial s} = -x \qquad y(\eta,0) = \eta,$$
$$\frac{\partial u}{\partial s} = 1 \qquad u(\eta,0) = 2.$$

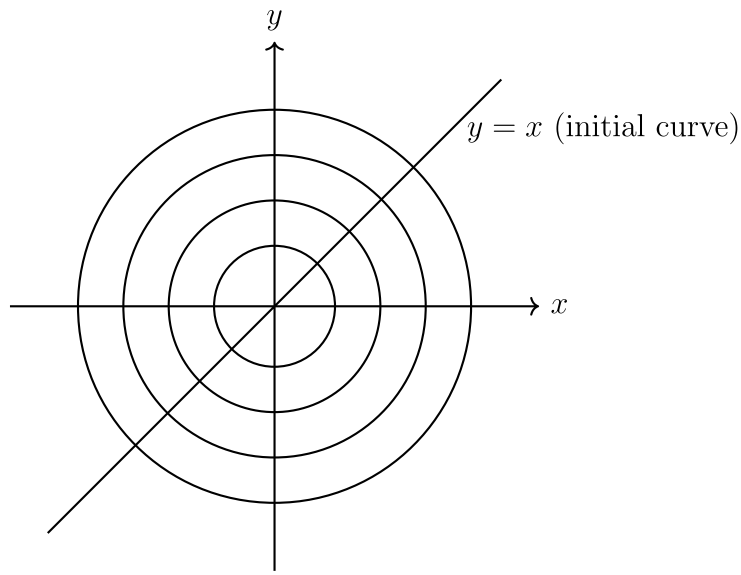
Now the first two ODEs are coupled. Before actually attempting to solve this system, let's see if we can figure out what the characteristics are. Remember that the characteristics are defined by the coefficients of the gradient components; in fact, the tangent vector at any point on a characteristic has slope equal to the ratio of the coefficients:

$$\frac{dy}{dx} = \frac{\partial y/\partial s}{\partial x/\partial s} = \frac{-x}{y}.$$

Perhaps to our dismay, this yields yet another differential equation, this time for the equation of our characteristics. This is first-order and separable, so we have

$$\int y\,dy = \int -x\,dx \implies \frac{y^2}{2} = -\frac{x^2}{2} + C(\eta) \implies x^2 + y^2 = \tilde{C}(\eta).$$

These characteristics are circles!



Integrating the last ODE and using the auxiliary condition gives $u(\eta, s) = s + 2$. Now, notice what happens if we differentiate the first ODE for x:

$$x_s = y \implies x_{ss} = y_s = -x \implies x_{ss} + x = 0. \tag{12}$$

This is a second order linear ODE with characteristic equation $r^2 + 1 = 0$. Thus the roots are $r = \pm i$ and, using the prototypical e^{rs} guess, we have that

$$\begin{aligned} x(\eta, s) &= c_1(\eta)\cos s + c_s(\eta)\sin s, \\ x(\eta, 0) &= c_1(\eta) = \eta \implies x(\eta, s) = \eta\cos s + c_s(\eta)\sin s. \end{aligned}$$

Now to find y, recall that it is simply the derivative of x in s:

$$\begin{aligned} y = x_s &= -\eta\sin s + c_2(\eta)\cos s \\ y(\eta, 0) &= \eta \implies c_2(\eta) = \eta. \end{aligned}$$

With both constants of integration known, we get the formulas for the original variables in terms of the characteristic parameters,

$$x = \eta(\cos s + \sin s)$$
$$y = -\eta(\cos s - \sin s).$$

and now all that is left to do is invert them to find $\eta(x, y)$ and $s(x, y)$.

2.6 Nonexistence of Solutions to First-Order PDEs

3 Appendix: Further Notes on the Method of Characteristics

4 The Wave Equation in One Space Dimension

5 Appendix: Review of Vector Calculus, Multiple Integrals, and the Divergence Theorem

6 The Wave Equation in Three and Two Space Dimensions

7 The Fourier Transform

8 The Diffusion Equation

9 Appendix: Further Notes on the Fourier Transform and the Heat Equation

10 The Laplacian, Laplace's Equation, and Harmonic Functions