Analysis I

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[1]These notes serve as an introduction to the analysis of functions of a real variable. Key results include the completeness of the real numbers, Cantor's theorem, the Cauchy Condensation Test, the Bolzano-Weierstrass theorem, the Heine-Borel theorem, the preservation of compact sets, the Extreme Value theorem, and the Generalized Mean Value theorem. Along the way, we will consider a host of interesting pathologies including the difference in cardinality between $\mathbb{Q}$ and $\mathbb{R}$, the Cantor set (an uncountable set with zero length having dimension $\log 2/\log 3$), the functions of Dirichlet and Thomae, and an everywhere-continuous nowhere-differentiable function.

Contents

[1]The content of these notes is largely based on *Understanding Analysis* by Stephen Abbott.

1 Introduction: What is Analysis?

The field of mathematics that has come to be known as **analysis** gets its name from the 1821 seminal text on advanced calculus written by Augustin-Louis Cauchy titled *Cours d'Analyse*. In it, Cauchy presents a rigorous treatment of differential and integral calculus that stresses the importance of axiomatic reasoning in preventing the logical pitfalls that previously pervaded the field. Euler, Lagrange, and the other calculus magnates of the 18th century worked under the fallacious assumption that algebraic manipulations that were valid in the finite case would extend without reservation to the infinite case. Cauchy calls this error in reasoning the **generalization of algebra** and, considering that calculus is the study of the infinite, he argues that a thorough and precise treatment of the logical underpinnings of calculus could prevent the kinds of errors made by his predecessors.

Example. Most calculus students treat infinite sums in exactly the same way as finite sums, but this generalization can lead to incredibly erroneous results. Take, for instance, the sum of all natural numbers

$$c = 1 + 2 + 3 + 4 + 5 + ...$$

Let's turn off our brains and see where the algebra takes us. Multiplying both sides by 4 and distributing produces

$$4c = 4 + 8 + 12 + 16 + 20 + ...$$

We can subtract this product from the original sum to obtain

$$c - 4c = (1 + 2 + 3 + 4 + 5 + ...) - (4 + 8 + 12 + 16 + 20 + ...),$$

and upon a little regrouping, we have

$$\begin{aligned} -3c &= 1 + (2 - 4) + 3 + (4 - 8) + 5 + (6 - 12) + ... \\ &= 1 - 2 + 3 - 4 + 5 - 6 + ... \end{aligned}$$

Now consider the power series

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + ...$$

At $x = 1$, this series becomes

$$\frac{1}{4} = 1 - 2 + 3 - 4 + 5 - 6 + ...$$

But this infinite sum is exactly the value of $-3c$! Because $1/4 = -3c$, we are forced to conclude that the sum of all natural numbers is

$$c = 1 + 2 + 3 + 4 + 5 + ... = -\frac{1}{12}.$$

Of course this is wrong, but it is hard to see where exactly the flaw in our reasoning occurs. The distribution of the scalar multiple and the commutativity of the terms of the infinite sum all come into question. The validity of the power series at $x = 1$ is also dubious. One objective of analysis is to prove which infinite sums may be manipulated by standard finite algebra, as well as to state the domains under which a power series is defined.

Example. Consider the doubly-indexed sequence

$$S_{m,n} = \frac{m}{m+n}.$$

A non-rigorous understanding of sequences might convince someone that the interchanging of limits is always valid, that is

$$\lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} = \lim_{m\to\infty} \lim_{n\to\infty} S_{m,n}.$$

However,

$$\lim_{n\to\infty} (\lim_{m\to\infty} S_{m,n}) = \lim_{n\to\infty} 1 = 1,$$
$$\lim_{m\to\infty} (\lim_{n\to\infty} S_{m,n}) = \lim_{m\to\infty} 0 = 0.$$

Using analysis, we will prove explicitly under which conditions this interchange is valid.

Example. If a function f is infinitely differentiable, we might expect the power series g given by its Taylor coefficients to have a non-trivial radius of convergence, i.e. g should converge to f on some interval, however small. But let's let f be given by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

and see what happens when we expand about the point $x = 0$. Some simple calculus reveals that $0 = f(0) = f'(0) = f''(0) = ...$ which implies that every Taylor coefficient is 0. Consequently, the Taylor series about $x = 0$ is the function $g \equiv 0$ which doesn't converge to f on any interval of $\mathbb{R}$.

Example. The Fundamental Theorem of Calculus seems to suggest that if F is differentiable on $[a, b]$, then its derivative F' would be integrable on $[a, b]$. Surprisingly, his is not always the case. Let $F : [-1, 1] \to \mathbb{R}$ be given by

$$F(x) = \begin{cases} x^2 \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then the derivative of F is

$$F'(x) = \begin{cases} 2x \cos \frac{1}{x^2} + \frac{2}{x} \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

But because $|2x \cos(1/x^2)| < 2$, and $(2/x) \sin(1/x^2)$ is unbounded, the derivative increases and decreases without bound as $x \to 0$ from either side. As a result, the integral of the derivative

$$\int_{-1}^{1} F'$$

does not exist.

Example. For a given sequence of differentiable functions f_n that converge to a differentiable f, we might be inclined to think that the derivatives f_n' converge to f'. But consider

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Taking the limit as $n \to \infty$, we see that these functions converge to the zero function $f \equiv 0$. But each function has derivative

$$f_n'(x) = \sqrt{n} \cos(nx),$$

and it is clear that f_n' does not approach $f' \equiv 0$ as $n \to \infty$.

Example. [2] As a final example of the logical inconsistencies of calculus, we'll look the Calc 3 student's favorite procedure: the interchanging of partial derivatives. Simply put, for a function $f(x, y)$, we expect to have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

[2]This is Example 1.2.11 in *Analysis I* by Terence Tao.

But consider the function

$$f(x,y) = \frac{xy^3}{x^2+y^2}$$

whose first partials are

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{3xy^2}{x^2+y^2} - \frac{2xy^4}{(x^2+y^2)^2},\\ \frac{\partial f}{\partial x}(x,y) &= \frac{y^3}{x^2+y^2} - \frac{2x^2y^3}{(x^2+y^2)^2}.\end{aligned}$$

To compute the mixed partials at the point $(x,y) = (0,0)$, we can consider each first derivative along a coordinate axis and take the remaining partial:

$$\begin{aligned}\frac{\partial f}{\partial y}(x,0) &= \frac{0}{x^2} - \frac{0}{x^4} = 0 \implies \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0,\\ \frac{\partial f}{\partial x}(0,y) &= \frac{y^3}{y^2} - \frac{0}{y^4} = y \implies \frac{\partial^2 f}{\partial y \partial x}(0,0) = 1.\end{aligned}$$

It seems as though the interchanging of partials is not always permitted, and we will have to discover exactly under which conditions it is permissible.

Analysis can be holistically defined as the study of functions of a continuous variable. These notes will focus on **real analysis** (the study of functions of a real variable), and our first objective will be to rigorously examine what exactly the real numbers are. We will use a nonconstructive approach, meaning we will not go through the trouble of building the reals from the rationals. Instead, we opt to anchor the existence of $\mathbb{R}$ on the **axiom of completeness**. Throughout the notes, we will prove various theorems which are logically equivalent to AoC, such as the Monotone Convergence theorem and (under the assumption of the Archimedean Property) the Nested Interval Property, the Bolzano-Weierstrass theorem, and Cauchy's Criterion. What's important is understand that *any* of these theorems could've been taken as the defining axiom of the real numbers after which the others would then be proven. After investigating various properties of $\mathbb{R}$, we will work towards a rigorous justification of the main theorems of calculus.

Pedagogically speaking, a typical three-semester analysis sequence consists of a semester of differential analysis, a semester of integral and multivariate analysis, and a semester of measure theory and abstract integration. What follows is a traditional first semester.

2 The Real Numbers

2.1 Discussion: The Irrationality of $\sqrt{2}$

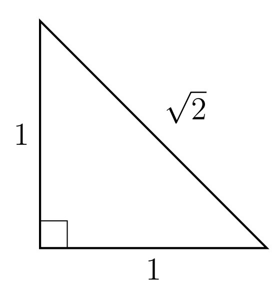
We begin with some familiar number systems.

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, ...\} \\
\mathbb{Z} &= \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \\
\mathbb{Q} &= \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\} \\
\mathbb{R} &= \ ?
\end{aligned}$$

These are the natural, integer, rational, and real number systems, respectively. As we can see, it is not terribly difficult to fully characterize the natural, integer, and rational numbers. In the case of the first two, we can simply write out the elements that belong to each set one-by-one. It is harder to write out the rationals in such a way (though possible as we will see), but nevertheless we are able to produce a simple rule in set-builder notation whereby each rational is defined. The real numbers prove to be insidious in this regard. How exactly do we define the real numbers? And why do we need numbers that are not rational?

The Pythagorean theorem provides a relationship between the lengths of the bases of a right triangle and the length of the hypotenuse. If the base sides have lengths a and b, then the hypotenuse has length $c = \sqrt{a^2 + b^2}$. Consider a right triangle with base sides of length 1. Then, by the Pythagorean theorem, the hypotenuse has length

$$c = \sqrt{1^2 + 1^2} = \sqrt{2}.$$



But as we will soon see, the square root of 2 is irrational. What are we to make of this? We would expect that the length of every measurable distance would be an actual number. The fact that the length of the hypotenuse is a non-rational number illustrates that there are a "larger" class of numbers for us to define. These are the so-called real numbers and they are the purpose of these notes. The following theorem was known to the Greeks.

Theorem 1. *There is no rational number whose square is 2.*

Remark. This is often stated as $\sqrt{2}$ is irrational, but this assumes $\sqrt{2}$ exists, which we also need to prove.

Proof (by contradiction). Suppose that there were two integers p and q, $q \neq 0$, such that $(\frac{p}{q})^2 = 2$. We may assume p and q have no common factors. Then $p^2/q^2 = 2$, or equivalently,

$$p^2 = 2q^2.$$

The following lemma will be used to show that p and q are both even, contradicting the assumption that they have no common factors.

Lemma. *Let n be an integer.*

(a) If n is even, then n^2 is even.

(b) If n is odd, then n^2 is odd.

Proof of lemma. The proof for each case depends on what it means for an integer to be even or odd.

(a) Suppose $n \in \mathbb{Z}$ and n is even. Then $n = 2k$ where $k \in \mathbb{Z}$. This implies that $n^2 = 4k^2 = 2(2k^2)$ is even.

(b) Suppose $n \in \mathbb{Z}$ and n is odd. Then $n = 2k + 1$ where $k \in \mathbb{Z}$. This implies that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd.

□

Let's use our new lemma to finish the proof of Theorem 1. We know that p^2 is even because $p^2 = 2q^2$. Therefore by the lemma, p is even. Writing $p = 2m$, $m \in \mathbb{Z}$, we find that $p^2 = 4m^2 = 2q^2$. But this means that $2m^2 = q^2$, from which it follows that q^2, and thus q, is even as well. This contradicts the assumption that p and q have no common factors. □

Since $\sqrt{2}$ can be the length of a line segment (e.g. the hypotenuse above), there is a "hole" in the rational number line precisely where this length is. The real number system remedies this by filling in the holes of the rational numbers.

Example. We will show that there is no rational number whose square is 3. Like the previous proof, it will be helpful to establish an intermediate lemma.

Lemma. *Let n be an integer.*

(a) If n is divisible by 3, then n^2 is divisible by 3.

(b) If n is not divisible by 3 (has remainder 1 or 2), then n^2 is not divisible by 3.

Proof of lemma. We will proceed, again, by cases.

(a) If n is divisible by 3, then $n = 3k$ for some integer k. Then $n^2 = 9k^2 = 3(3k^2)$ is divisible by 3.

(b) Case 1: If $n = 3k + 1$ for some integer k, then $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ is not divisible by 3.
Case 2: If $n = 3k+2$ for some integer k, then $n^2 = 9k^2+12k+4 = 9k^2+12k+3+1 = 3(3k^2 + 4k + 1) + 1$ is not divisible by 3.

□

We can restate the lemma as follows: for an integer n, n^2 is divisible by 3 if and only if n is divisible by 3. Now assume that $(\frac{p}{q})^2 = 3$ where p and $q \neq 0$ are integers with no common factor. Then $p^2 = 3q^2$, which implies that p^2 is divisible by 3. The lemma shows us that p is also divisible by 3, for which we can write $p = 3k$ for some integer k. Now,

$$p^2 = 3q^2 \implies (3k)^2 = 3q^2 \implies 9k^2 = 3q^2 \implies 3k^2 = q^2.$$

But since q^2 is divisible by 3, it follows that q is itself divisible by 3. This contradicts the assumption that p and q have no common factors and we are forced to conclude that there is no rational number whose square is 3.

2.2 Some Preliminaries

2.2.1 Notions from logic

Let A and B denote statements. Two statements can be joined through implication, written $A \implies B$, and read "A implies B" (alt. "if A, then B"). The implication $A \implies B$ means one of the following:

(i) A is true and B is true, or

(ii) A is false.

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n ends with 4 when written in base 10" and B the statement "n is even." Clearly $A \implies B$ because all numbers ending in 4 are even. Does $B \implies A$? The answer is no, because there exist even numbers that don't end in 4, such as 12. This lack of implication may be written $B \not\Longrightarrow A$.

When an implication is bi-directional, we write $A \iff B$, read "A is equivalent to B" (alt. "A if and only if (iff) B"). This is used to show that two statements are logically equivalent. The equivalence $A \iff B$ means one of the following:

(i) A and B are both true, or

(ii) A and B are both false.

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n ends with a 0, 2, 4, 6, or 8 when written in base 10" and B the statement "n is even." Then $A \iff B$.

Another way of stating the implication $A \implies B$ is to say that A only occurs if B occurs. The truth of A is contingent on B, in other words. The implication may then be read "A only if B." This is not to say that *every* time B occurs, A must as well (that would be "A if B"). This is the subtle difference between "A if B" ($B \implies A$) and "A only if B" ($A \implies B$).

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n is a multiple of 6" and B the statement "n is a multiple of 3." Then $A \implies B$. Here the phrase "if A, then B" means that if a number n is a multiple of 6, then n is also a multiple of 3. The equivalent phrase "A only if B" means that a number n is a multiple of 6 **only if** n is also a multiple of 3.

The contrapositive of an implication is equivalent to the implication itself. Thus, if $A \implies B$, then $\neg B \implies \neg A$. Why is this true? Well $A \implies B$ is true iff (A, B are both true) or (A is false). The contrapositive $\neg B \implies \neg A$ is true iff ($\neg B$, $\neg A$ are both true) or ($\neg B$ is false), which is equivalent to (B, A are both false) or (B is true). Every truth assignment that makes the original implication true makes the contrapositive true, and vice versa. Thus the two statements are logically equivalent.

2.2.2 Convention regarding definitions

Nearly all mathematical definitions are stated with "if" when "iff" is meant. For example, the definition below really means that a number is prime if and only if it is not 1 and has no factors other than 1 and itself.

Definition. A natural number $n \neq 1$ is **prime** if n has no factors other than 1 and n.

2.2.3 Quantifiers

The introduction of certain quantifiers will aid in the mathematical discussions that follow. The symbol $\forall$ is read "for all" and means that a statement holds for all elements in the domain of inquiry. For example, the statement "$\forall n \in \mathbb{N}\ ,\ \ n < n+1$" means that every natural number is less than the number after it. It is read "for all natural numbers n, n is less than $n+1$." This can be verified by adding n to both sides of the inequality $0 < 1$. The symbol $\exists$ is read "there exists" and means that a statement holds for at least one element in the domain of inquiry. For example, the statement "$\exists x \in \mathbb{Q}$ such that $4.1 < x < 4.2$" means that there exists a rational number that is greater than 4.1 and less than 4.2. This can be verified by simply providing an x that satisfies this property, namely $x = 4.15$.

We may speak of the negation of a proposition A, written $\neg A$, whose truth value is opposite that of A's. For instance, the negation of

$$\exists\ \frac{p}{q} \in \mathbb{Q} \text{ such that } \left(\frac{p}{q}\right)^2 = 2$$

is

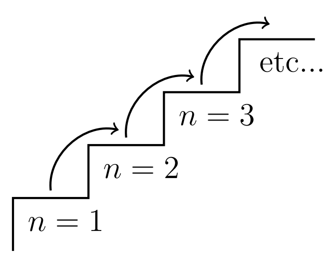
$$\nexists\ \frac{p}{q} \in \mathbb{Q} \text{ such that } \left(\frac{p}{q}\right)^2 = 2,$$

or equivalently,

$$\forall\ \frac{p}{q} \in \mathbb{Q},\ \left(\frac{p}{q}\right)^2 \neq 2.$$

2.2.4 Proof by induction

Suppose that for each natural number n, $S(n)$ is a statement involving n. The goal of proof by induction is to prove that the statements $S(n)$ are true for all $n \in \mathbb{N}$. The main idea is to begin by showing $S(1)$ is true, i.e. that the statement is true for the first natural number. This is often referred to as the base case. Next, we want to show that if the statement is true for a generic natural n, then it is also true for $n+1$, i.e. that $S(n) \implies S(n+1)$. This is referred to as the inductive step.



Example. Let $y_1 = 1$ and for each $n \in \mathbb{N}$, define $y_{n+1} = \frac{3y_n+4}{4}$. We will use induction to prove that the sequence satisfies $y_n < 4$ for all natural numbers n. More specifically, induction will be used to show that $S(n) =$ "$y_n < 4$" is true for all $n \in \mathbb{N}$.
Base case: For $n = 1$, we have $y_1 = 1 < 4$ and the statement $S(1)$ is true.
Inductive step: Assume that $S(n)$ is true, i.e. that $y_n < 4$. We must show that $S(n+1)$ is also true, i.e. that $y_{n+1} < 4$. Multiplying both sides of the given inequality by 3 yields $3y_n < 12$. A little more algebra produces

$$\frac{3y_n + 4}{4} < \frac{12 + 4}{4} = \frac{16}{4} = 4.$$

But the LHS of this inequality is precisely y_{n+1}, and so we have that $y_{n+1} < 4$ and the statement is true for $S(n+1)$. By the principle of induction, $y_n < 4$ for all natural numbers.

Example. Mathematical induction can be used to show that $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.
Base case: For $n = 1$, we have $4^n - 1 = 4 - 1 = 3$ is divisible by 3.
Inductive step: Assume $4^n - 1$ is divisible by 3, i.e. $4^n - 1 = 3k$ for some integer k. Then $4^n = 3k + 1$. Multiplying both sides by 4 produces $4^{n+1} = 12k + 4$, from which we have

$$4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$$

is divisible by 3.

Example. Let $y_1 = 1$ and for each $n \in \mathbb{N}$, define $y_{n+1} = \frac{3y_n+4}{4}$. We will use induction to prove that the sequence

$$(y_n)_{n=1}^{\infty} = (y_1, y_2, y_3, ...)$$

is strictly increasing. This amounts to showing that $y_n < y_{n+1}$ for all natural n.
Base case: For $n = 1$, we must show that $y_1 < y_2$. We have that $y_1 = 1$ and

$$y_2 = \frac{3y_1 + 4}{4} = \frac{3 + 4}{4} = \frac{7}{4}.$$

Since $1 < \frac{7}{4}$, the base case is true.
Inductive step: Suppose $y_n < y_{n+1}$. We must show that $y_{n+1} < y_{n+2}$. Behold:

$$y_n < y_{n+1} \implies 3y_n < 3y_{n+1} \implies 3y_n + 4 < 3y_{n+1} + 4 \implies \frac{3y_n + 4}{4} < \frac{3y_{n+1} + 4}{4}$$
$$\implies y_{n+1} < y_{n+2}.$$

2.3 The Axiom of Completeness

We are ready to begin our discussion of the real numbers. The real number system $\mathbb{R}$ is an ordered field containing the rational numbers equipped with the familiar operations of addition and multiplication from $\mathbb{Q}$. Moreover, the field $\mathbb{R} \supseteq \mathbb{Q}$ satisfies the Axiom of Completeness. This is the defining difference between $\mathbb{R}$ and $\mathbb{Q}$—it is precisely this axiom that prevents the existence of holes and gaps along the real number line.

But before discussing this seminal axiom, let's review what it means for $\mathbb{R}$ to be a field. The fact that $\mathbb{R}$ is a field means that the following field axioms are satisfied for all $a, b, c \in \mathbb{R}$:

F1. $a + b = b + a$ (commutativity of addition)

F2. $a + (b + c) = (a + b) + c$ (associativity of addition)

F3. $a + 0 = 0 + a = a$ (existence of an additive identity)

F4. $a + (-a) = 0$ (existence of additive inverses)

F5. $ab = ba$ (commutativity of multiplication)

F6. $a(bc) = (ab)c$ (associativity of multiplication)

F7. $a \cdot 1 = 1 \cdot a = a$ (existence of a multiplicative identity)

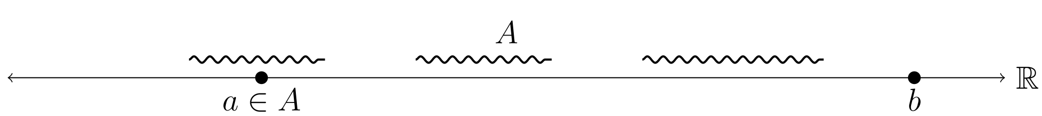
F8. $a \cdot \frac{1}{a} = 1$ (existence of multiplicative inverses)

F9. $a(b + c) = ab + ac$ (distributive property)

F10. $0 \neq 1$ (difference between identities).

Further, $\mathbb{R}$ is an ordered field, meaning there is a relation called $<$ defined on pairs of real numbers. More concretely, for all $x, y \in \mathbb{R}$, either $x < y$, $y < x$, or $x = y$, with one and only one of these relations holding. This ordering is transitive; if $x, y, z \in \mathbb{R}$, then $x < y$ and $y < z$ imply that $x < z$.

Definition. A subset $A \subseteq \mathbb{R}$ is **bounded above** if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Such a number b is called an **upper bound** of the set A. Trivially, if $A = \emptyset$, then any $b \in \mathbb{R}$ is an upper bound for A.



Example. Let $A = (1,5) = \{x \in \mathbb{R} : 1 < x < 5\}$. Then some upper bounds of A include 76, 10, and 5.

Example. Let $A = [1,5] = \{x \in \mathbb{R} : 1 \leq x \leq 5\}$. Then 76, 10, and 5 are still upper bounds of A despite 5 being an element of A.

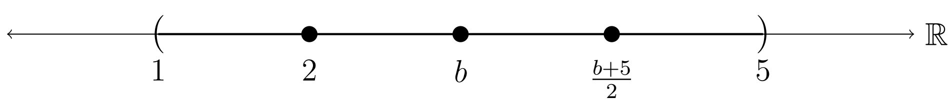
Definition. If $A \subseteq \mathbb{R}$ is bounded above, then an upper bound c of A is a **least upper bound (supremum)** of A if $c \leq b$ for every upper bound b of A.

That this upper bound is unique is immediate, and we will write $\sup A = c$.

Example. We would like to find the least upper bound of $A = (1,5)$. A reasonable candidate is 5, which is obviously an upper bound because $x < 5$ for all $x \in A$. But we can prove it is the *least* upper bound.

Claim. There is no upper bound of A smaller than 5.

Proof. Proceed by contradiction: suppose there exists an upper bound b of the set A such that $b < 5$. By the definition of upper bound, we have that b is greater than or equal to every element of A; namely $b \geq 2$ because $2 \in A$. But, if we find just one element in A that is greater than b, then our assumption fails and the claim is proven. Based on the geometry of the number line below, the element $\frac{b+5}{2}$ looks to be a good candidate for this type of counterexample.



For this counterexample to work, we must show two things. Firstly, this number has to actually be an element of the set $A = (1,5)$, that is, we must show

$$1 < \frac{b+5}{2} < 5.$$

This can be accomplished by deriving the inequalities as follows:

$$2 \leq b \implies 1 \leq \frac{b}{2} < \frac{b+5}{2} \implies 1 < \frac{b+5}{2},$$

$$b < 5 \implies \frac{b}{2} < \frac{5}{2} \implies \frac{b}{2} + \frac{5}{2} < \frac{5}{2} + \frac{5}{2} \implies \frac{b+5}{2} < 5.$$

Secondly, this element must be greater than the proposed upper bound b. This, too, can be shown through a manipulation of the given inequality $b < 5$:

$$b < 5 \implies \frac{b}{2} < \frac{5}{2} \implies \frac{b}{2} + \frac{b}{2} < \frac{b}{2} + \frac{5}{2} \implies b < \frac{b+5}{2}.$$

This is all we needed to show. By assuming that an upper bound less than 5 existed, we reached a logical contradiction. The only thing left to conclude is that the supremum of A is indeed 5. $\square$

With these definitions in hand, we can now state the axiom at the heart of this section.

Axiom of Completeness. Every non-empty set of real numbers that is bounded above has a least upper bound.

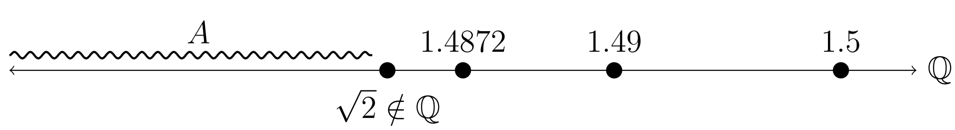
We claimed earlier that this axiom is what was needed to mend the holes present in the rational number system. We also showed earlier that there is no rational number whose square is 2. Consider the set

$$A = \left\{x \in \mathbb{Q} \mid x^2 < 2\right\}.$$

The rational number system is not complete, and this fact can be demonstrating by showing that A has no least upper bound. For every upper bound of A, we can find another upper bound smaller than it. Knowing that $\sqrt{2} = 1.487121...$, some successively smaller upper bounds of A include $1.5, 1.49, 1.488, 1.4872, 1.48713, 1.487122, ...$ but this sequence will both never end and never reach $\sqrt{2}$.

The completeness of the real numbers is exactly what's needed to give A a supremum. In fact, in the next section we will be able to prove that

$$\sup A = \sqrt{2}.$$



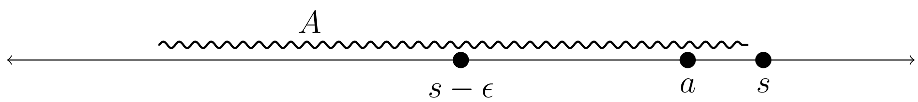
Lemma. *Assume $s \in \mathbb{R}$ is an upper bound for a non-empty set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for every $\epsilon > 0$, there is an element $a \in A$ such that $s - \epsilon < a$.*

Proof. ($\implies$) : Suppose $s = \sup A$ and let $\epsilon > 0$ be given. Then $s - \epsilon < s$. Since s is the *least* upper bound, $s - \epsilon$ is not an upper bound. Because of this, there exists an element $a \in A$ such that $a > s - \epsilon$.

$(\Longleftarrow)$: Suppose that for every $\epsilon > 0$ there exists an element $a \in A$ such that $s - \epsilon < a$. We want to show that s is the least upper bound of A, equivalently, that if t is any generic upper bound of A then $s \leq t$. We'll suppose for the sake of contradiction that there exists an upper bound $t < s$. We can write

$$t = s - (s - t) = s - \epsilon$$

where $\epsilon := s - t > 0$. By assumption, there exists an element $a \in A$ such that $t = s - \epsilon < a$. But this contradicts t being an upper bound, and we are forced to conclude that $t \geq s$. Because s is less than or equal to all upper bounds of A, it must be that $\sup A = s$. □



Example. We would like to find the supremum of the set

$$A = \{2x + 3 \mid x \in \mathbb{Q} \text{ and } 6 < x < 7\}$$

and prove its validity. A reasonable candidate for the least upper bound is $\sup A = 17$. At the very least, it is **an** upper bound because

$$x < 7 \implies 2x + 3 < 2 \cdot 7 + 3 = 17.$$

For the sake of contradiction, suppose b is an upper bound and $b < 17$. It will help to define the set

$$B = \{x \in \mathbb{Q} \text{ and } 6 < x < 7\}.$$

Because b is an upper bound, $2x + 3 < b$ for all $x \in B$. A little algebra reveals

$$2x < b - 3 \implies x < \frac{b-3}{2} \text{ for all } x \in B,$$

and if $b < 17$, then

$$\frac{b-3}{2} < \frac{17-3}{2} = 7.$$

But between any two real numbers there exists a rational, so we can conclude that there must be an element $x \in B$ such that

$$\frac{b-3}{2} < x < 7$$

which is a contradiction. There are no upper bounds less than 17; consequently, $\sup A = 17$.

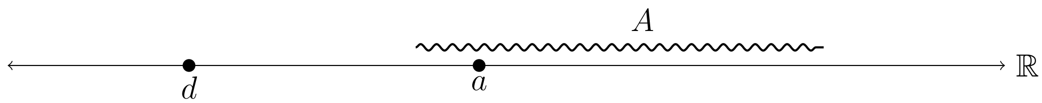
The crux of the proceeding argument rests upon the perhaps not-so-obvious fact that between any two real numbers lies a rational number. This should not have been used without justification and in the next section we shall prove this property of the rationals, called the **density of $\mathbb{Q}$ in $\mathbb{R}$**.

2.4 Consequences of Completeness

2.4.1 Infimums

One might wonder if sets of real numbers that are bounded below admit greatest lower bounds. The answer is yes, and it can be shown from the Axiom of Completeness.

Definition. If $A \subseteq \mathbb{R}$, then d is a **lower bound** for A if $d \leq a$ for all $a \in A$.



Definition. If d is a lower bound for A such that $c \leq d$ for all lower bounds c of A, then d is the **greatest lower bound (infimum)** of A and we write $\inf A = d$.

Claim. If A is a non-empty set of real numbers that is bounded below, then there exists a greatest lower bound of A.

Not only does a proof of this rely on the Axiom of Completeness, the claim is actually logically equivalent to AoC itself. Therefore, one could have taken this claim to be the defining axiom of the real numbers by which the existence of suprema for upper-bounded sets would be established as a result.

2.4.2 The Density of $\mathbb{Q}$ in $\mathbb{R}$

The next two theorems tells us how $\mathbb{N}$ and $\mathbb{Q}$ fit inside of $\mathbb{R}$.

Theorem 2 (Archimedean Property).

(i) *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > x$.*

(ii) *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ such that $1/n < y$.*

Proof. **(i)** If this statement were false, then that would mean the natural numbers are bounded from above. So suppose there exits an $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. Then x is an upper bound for $\mathbb{N}$. By AoC, there exists a least upper bound $c = \sup \mathbb{N}$. But then $c - 1$ is not an upper bound of $\mathbb{N}$, meaning there exits an $n_0 \in \mathbb{N}$ such that $n_0 > c - 1$. This is equivalent to $n_0 + 1 > c$. But since $n_0 + 1 \in \mathbb{N}$, this contradicts c being an upper bound of $\mathbb{N}$.

(ii) Let $y > 0$. Because

$$y \cdot \frac{1}{y} = 1,$$

it must be the case that $1/y > 0$ (or else the above would not hold). By (i), there exists an $n \in \mathbb{N}$ such that $1/y < n$. Multiplying both sides of the inequality by y/n produces

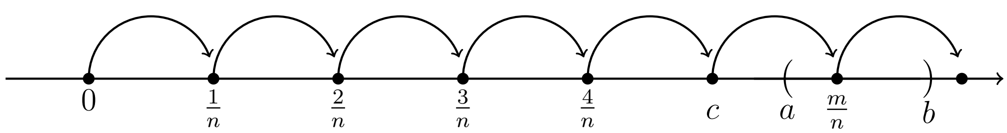
$$\frac{1}{y} \cdot \frac{y}{n} < n \cdot \frac{y}{n} \implies \frac{1}{n} < y.$$

□

Theorem 3 (**Density of $\mathbb{Q}$ in $\mathbb{R}$**)**.** *For every two real numbers a and b with $a < b$, there exists an $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that*

$$a < \frac{m}{n} < b.$$

Proof. The trick is to let the denominator n of our rational number be small enough to fit inside the interval (a, b) whose length is $b - a$. By the Archimedean Property, there exists an $n \in \mathbb{N}$ such that $1/n < b - a$. Using $1/n$ as the size of our "ruler", we count how many ruler measurements we need to get from the origin to the inside of the interval. The number of ruler measurements will be m.



We'll look at the set of ruler measurements that are left of the interval. Let

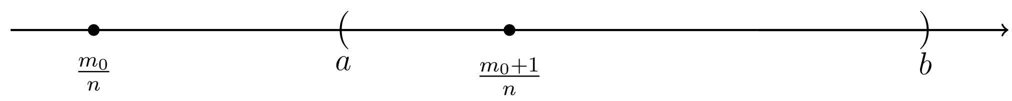
$$A = \left\{ \frac{m}{n} \;\middle|\; m \in \mathbb{Z} \text{ and } \frac{m}{n} \leq a \right\}.$$

Then a is an upper bound of A, and A must have a supremum. Call $c = \sup A$. Then $c - 1/n$ is **not** an upper bound of A. Thus there exists an element of A, say $m_0/n \leq a$, such that

$$c - \frac{1}{n} < \frac{m_0}{n} \leq a.$$

This is the last ruler measurement to the left of the interval. So add one more "measure" of 1/n:

$$c < \frac{m_0}{n} + \frac{1}{n} = \frac{m_0 + 1}{n} \notin A.$$



The measurement $(m_0 + 1)/n$ is not in the set A because it is greater than $c = \sup A$. Consequently,

$$\frac{m_0 + 1}{n} > a.$$

To prove it lies in the interval (a, b), we must also show it is less than b. To this end, notice that

$$\frac{m_0 + 1}{n} = \frac{m_0}{n} + \frac{1}{n} \leq a + \frac{1}{n} < a + (b - a) = b \implies \frac{m_0 + 1}{n} < b.$$

Because $a < (m_0 + 1)/n < b$, we have found a rational between the reals. □

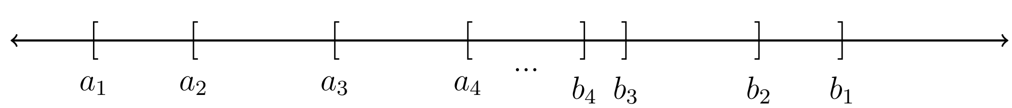
2.4.3 Nested Closed Intervals

Consider a sequence of closed intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ..., $I_n = [a_n, b_n]$, ... such that

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_n \supseteq ...$$

The fact that I_n is an interval means that $a_n \leq b_n$ for all $n \in \mathbb{N}$. The fact that the intervals are nested means that

$$a_1 \leq a_2 \leq a_3 \leq ...$$
$$b_1 \geq b_2 \geq b_3 \geq ...$$



Example. Fix a choice of $m \in \mathbb{N}$. We can show that $a_n \leq b_m$ for all $n \in \mathbb{N}$. There are three cases to consider: $n = m$, $n < m$, or $n > m$.

1. If $n = m$, then $a_m \leq b_m$ because $I_m = [a_m, b_m]$ is an interval.

2. If $n < m$, then $a_n \leq a_m \leq b_m$ which implies that $a_n \leq b_m$.

3. If $n > m$, then $a_n \leq b_n \leq b_m$ which implies that $a_n \leq b_m$.

Theorem 4 (Nested Interval Property). *The nested intersection of closed intervals*

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_n \supseteq ...$$

has a nonempty intersection, i.e. $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. □

2.4.4 Existence of Square Roots

2.5 Cardinality

3 Sequences and Series

4 Basic Topology of $\mathbb{R}$

5 Functional Limits and Continuity

6 The Derivative

7 Sequences and Series of Functions